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On Nonlinear Uniform Approximation

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1. INTRODUCTION

Let $C[a, b]$ be the normed linear space consisting of all real-valued continuous functions defined on the closed interval $[a, b]$ $(a < b)$ with the uniform norm $\|\psi\| = \max\{|\psi(x)|: x \text{ in } [a, b]\}$. Further, let $\mathscr{F}$ be a nonempty subset of $C[a, b]$. For an arbitrary element, ϕ, in $C[a, b]$, an element F in $\mathscr{F}$ is said to be a best approximation from $\mathscr{F}$ to ϕ if $\|F - \phi\| \leq \|G - \phi\|$ for all G in $\mathscr{F}$. The set $\mathscr{F}$ is called an approximating family of functions on $[a, b]$. In order to obtain useful answers to questions about existence, uniqueness, and characterization of best approximations it has been necessary to consider special approximating families.

In defining a varisolvent family $\mathscr{F}$ in 1961 (e.g., see [8]), Rice extracted the properties of polynomials which were useful in the development of the linear theory. At this time a fairly complete theory of varisolvent families in the sense of Rice exists. Although Rice's definition of varisolvency allows as a special case the family of exponential sums, it does not include the much studied family of generalized exponential sums (sums of exponentials with polynomial coefficients).

Motivated by Rice's definition of varisolvency and by the intriguing alternation theorem for generalized exponential sums given by Braess [2], we have attempted to define a class of nonlinear families of approximating functions which includes both of the above families. At the risk of confusion, we call families of this class varisolvent families.

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2. Definition of a Varisolvent Family

In this section, $\mathscr{F}$ will be a subset of the continuous real-valued functions $[a, b]$ and $\| \; \|$ will denote the maximum norm on $C[a, b]$. A degree of a function in $\mathscr{F}$ will be assigned if it possesses certain properties relative to the family $\mathscr{F}$. After some preliminary lemmas are presented, the notion of a varisolvent family as an approximating family of functions is introduced. In general, when approximating a continuous function by elements of a varisolvent family, one is not guaranteed either the existence or the uniqueness of a best approximation. However, an alternation theorem is given which characterizes best approximations from a varisolvent family.

First, some definitions are needed.

Definition 2.1. Let $\{I_i\}_{i=1}^{n}$ be a sequence of closed intervals ($n \geqslant 1$). The sequence $\{I_i\}_{i=1}^{n}$ will be called an *increasing sequence of closed intervals* if for every x in I_i and every y in I_{i+1} ($1 \leqslant i < n$), it it srue that $x < y$.

Definition 2.2. Let ψ be a continuous real-valued nonzero function on $[a, b]$. The function ψ is said to *alternate n times* ($n \geqslant 0$) on $[a, b]$ if there is an increasing set of points $\{x_i\}_{i=1}^{n+1}$ in $[a, b]$ such that $\| \psi \| = | \psi(x_i)|$ ($i = 1,..., n + 1$) and $\psi(x_i)\, \psi(x_{i+1}) < 0$ ($1 \leqslant i < n + 1$). The increasing set of points $\{x_i\}_{i=1}^{n+1}$ ($n \geqslant 0$) that satisfy the above is called a *set of alternation points for* ψ.

Definition 2.3. Let $\mathscr{F}$ be a family of functions in $C[a, b]$ and let F be in $\mathscr{F}$. The ordered pair of integers (n_1, n_2) with $n_1 \geqslant 0$ and $n_2 \geqslant 1$ is *a degree of F with respect to $\mathscr{F}$* if the following conditions are met:

(1) Let $\epsilon > 0$ and σ in $\{-1, 1\}$ be arbitrarily chosen. If $n_1 = 1$, then there is a function, G, in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)(F(x) - G(x)) > 0$ on $[a, b]$. (The factor (-1) is superfluous for this part of the definition.) If $n_1 > 1$, if δ is an arbitrary element of $\{0, 1\}$, and if $\{[c_i, d_i]\}_{i=1}^{n_1-\delta}$ is an arbitrary increasing sequence of closed intervals where $c_1 = a$ and $d_{n_1-\delta} = b$, then there is a function, G, in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ ($i = 1,..., n_1 - \delta$).

(2) If G is a continuous function on $[a, b]$ and $a \leqslant x_1 < \cdots < x_{n_2+1} \leqslant b$ such that $(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) < 0$ ($i = 1,..., n_2$), then G is not in the family $\mathscr{F}$.

It is noted that $n_1 = 0$ is permissible and that if $(0, n_2)$ is a degree of F with respect to $\mathscr{F}$, only the integer n_2 gives any information about the function's relation to the rest of the family. Furthermore, if F has (n_1, n_2) as a degree, $(0, n_2)$ is also a degree.

What the above definition is saying is that if the function, F, in $\mathscr{F}$ has (n_1, n_2) as a degree with respect to $\mathscr{F}$, then there is a function G in $\mathscr{F}$ that is arbitrarily close to F on $[a, b]$ such that $F - G$ alternates in sign on n_1 $(n_1 - 1)$ intervals. Furthermore, every member of $\mathscr{F}$ that is distinct from F crosses F at most $n_2 - 1$ times in (a, b). If an approximating family, $\mathscr{F}$, satisfies Rice's definition of Property A [8], then the first part of Definition 2.3 would be satisfied, but the converse is not necessarily true.

Remark 2.4. If (n_1, n_2) is a degree of F with respect to $\mathscr{F}$, then $n_1 \leq n_2$. To see this, assume $n_2 < n_1$ (thus $n_1 > 1$). Let $\epsilon = 1$, $\sigma = 1$, $\delta = 0$, and $\{[c_i, d_i]\}_{i=1}^{n_1}$ be an increasing sequence of closed intervals ($i = 1,\ldots, n_1$) with $c_1 = a$ and $d_{n_1} = b$. Because (n_1, n_2) is a degree of F, there is a G in $\mathscr{F}$ such that $(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ ($i = 1,\ldots, n_1$). Let $x_i = \frac{1}{2}(c_i + d_i)$, which is in $[c_i, d_i]$ ($i = 1,\ldots, n_1$). Since $(-1)^i(F(x_i) - G(x_i)) > 0$ ($i = 1,\ldots, n_1$), we have $(-1)(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) > 0$ ($i = 1,\ldots, n_1 - 1$). But since $n_1 - 1 \geq n_2$, we have $(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) < 0$ ($i = 1,\ldots, n_2$). This implies G is not in $\mathscr{F}$, which is a contradiction.

The definition seems to indicate that a function is permitted to have more than one degree. This is, in fact, the case. If F has a degree (n_1, n_2) with respect to $\mathscr{F}$, the following lemma gives some information as to what other degrees F may have.

LEMMA 2.5. *If F belongs to $\mathscr{F}$ and has degree (n_1, n_2) with respect to $\mathscr{F}$, then*

(1) *$(n_1 - 1, n_2)$ is also a degree of F with respect to $\mathscr{F}$ as long as n_1 is not zero or three;*

(2) *$(n_1, n_2 - 1)$ is also a degree of F with respect to $\mathscr{F}$.*

Proof. (1) In the case where $n_1 = 1$ or 2, the result follows immediately from the definition. Let $n_1 > 3$. Let positive ϵ, σ in $\{-1, 1\}$ and δ in $\{0, 1\}$ be chosen arbitrarily. Let $\{[c_i, d_i]\}_{i=1}^{n_1-1-\delta}$ be an arbitrary increasing sequence of closed intervals where $c_1 = a$ and $d_{n_1-1-\delta} = b$. If $\delta = 0$, then there exist G in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ ($i = 1,\ldots, n_1 - 1$) since (n_1, n_2) is a degree of F. If $\delta = 1$, let $\mu = \frac{1}{5}(c_{n_1-2} - d_{n_1-3}) > 0$. Define $J_i = [c_i, d_i]$ ($i = 1,\ldots, n_1 - 3$), $J_{n_1-2} = [d_{n_1-3} + \mu, d_{n_1-3} + 2\mu]$, $J_{n_1-1} = [d_{n_1-3} + 3\mu, d_{n_1-3} + 4\mu]$, $J_{n_1} = [c_{n_1-2}, d_{n_1-2}]$. Since (n_1, n_2) is a degree of F, there exist G in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on J_i ($i = 1,\ldots, n_1$). Thus $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ ($i = 1,\ldots, n_1 - 3$) and $\sigma(-1)^{n_1}(F(x) - G(x)) = \sigma(-1)^{n_1-2}(F(x) - G(x)) > 0$ on $[c_{n_1-2}, d_{n_1-2}]$.

(2) The proof follows immediately from the definition.

To illustrate why $n_1 \neq 3$ in Lemma 2.5 we now construct $\mathscr{F}$, a family of functions on $[a, b]$ such that the zero function belongs to $\mathscr{F}$ and the zero function has (3, 3) as a degree, while (1, 3) and (2, 3) are not degrees. Let $\mathscr{F}$ denote the set of all functions of the form $c(x - x_1)(x - x_2)$ or $c(x - x_1)$ where c is an arbitrary real constant and the x_i's are distinct values in the open interval (a, b).

An immediate consequence of Lemma 2.5 is the following corollary.

COROLLARY 2.6. *If F in $\mathscr{F}$ has a degree (n_1, n_2) with respect to $\mathscr{F}$ and $n_1 \geqslant 3$, then (m_1, m_2) is also a degree where $3 \leqslant m_1 \leqslant n_1$ and $n_2 \leqslant m_2 < \infty$.*

If F in $\mathscr{F}$ is an approximation from $\mathscr{F}$ to a continuous real-valued function, ϕ, the next four lemmas give sufficient conditions for the existence of an approximation to ϕ that is better than F.

LEMMA 2.7. *Let F in $\mathscr{F}$ have a degree (n_1, n_2) with $n_1 \geqslant 2$ and let ϕ belong to $C[a, b]$. If $F - \phi$ alternates $n_1 - 1$ times and does not alternate n_1 times, then there is a function, G, in $\mathscr{F}$ such that $\| G - \phi \| < \| F - \phi \|$.*

Proof. Let $\{x_i\}_{i=1}^{n_1}$ be a set of alternation points for $F - \phi$ in $[a, b]$. Define $x_0 = a$, $x_{n_1+1} = b$,

$$x_i^L = \min\{x \in [x_{i-1}, x_i]: (F(x) - \phi(x)) = (F(x_i) - \phi(x_i))\}$$

and

$$x_i^U = \max\{x \in [x_i, x_{i+1}]: (F(x) - \phi(x)) = (F(x_i) - \phi(x_i))\} \quad (i = 1,..., n_1).$$

The point $x_i^L(x_i^U)$ does exist since the set of which we are taking the minimum (maximum) of is compact, and $F - \phi$ is continuous.

Claim. $x_i^U < x_{i+1}^L$ $(i = 1,..., n_1 - 1)$. Indeed this is true, since $(F(x_i^U) - \phi(x_i^U)) = (F(x_i) - \phi(x_i))$, $(F(x_{i+1}^L) - \phi(x_{i+1}^L)) = (F(x_{i+1}) - \phi(x_{i+1}))$ and $(F(x_i) - \phi(x_i))(F(x_{i+1}) - \phi(x_{i+1})) < 0$, it follows that $(F(x_i^U) - \phi(x_i^U)) \times (F(x_{i+1}^L) - \phi(x_{i+1}^L)) < 0$ and thus $x_i^U \neq x_{i+1}^L$. Continuing in the proof of the claim, suppose $x_{i+1}^L < x_i^U$. We also have $x_i < x_{i+1}^L < x_i^U < x_{i+1}$. Define $\{y_i\}_{i=1}^{n_1+2}$ such that $y_j = x_j$ $(j = 1,..., i)$, $y_{i+1} = x_{i+1}^L$, $y_{i+2} = x_i^U$, and $y_j = x_{j-2}$ $(j = i + 3,..., n_1 + 2)$. Thus $F - \phi$ alternates at least $n_1 + 1$ times and hence alternates n_1 times which contradicts the assumption that $F - \phi$ does not alternate n_1 times. Therefore the claim is true. Define $\mu = \frac{1}{3}\min\{x_{i+1}^L - x_i^U: i = 1,..., n_1 - 1\}$ and define $I_1 = [a, x_1^U + \mu]$, $I_i = [x_i^L - \mu, x_i^U + \mu]$ $(1 < i < n)$ and $I_{n_1} = [x_{n_1}^L - \mu, b]$. Because of the way μ is defined, $\{I_i\}_{i=1}^{n_1}$ is an increasing sequence of closed intervals. Select σ in $\{-1, 1\}$ such that $\sigma(-1)(F(x_1) - \phi(x_1)) = \| F - \phi \|$. It follows from the definition of the intervals $[x_i^L, x_i^U]$ $(i = 1,..., n_1)$ that ϵ_1 is a positive number, where $\epsilon_1 = \min_{i=1,\ldots,n_1} \min\{\|F - \phi\| - \sigma(-1)^i(F(x) - \phi(x)): x \text{ in } I_i\}$. A short continuity argument will show that $\sup\{|F(x) - \phi(x)|: x \text{ in } [a, b] - \bigcup_{i=1}^{n_1} I_i\} < \|F - \phi\|$, therefore ϵ_2, defined as $\epsilon_2 = \|F - \phi\| - \sup\{|F(x) - \phi(x)|: x \text{ in } [a, b] - \bigcup_{i=1}^{n_1} I_i\}$, is positive. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Since (n_1, n_2) is a degree of F, there exist G in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on I_i $(i = 1,\ldots, n_1)$.

Now we show that $\|G - \phi\| < \|F - \phi\|$. It suffices to show that $|G(x) - \phi(x)| < \|F - \phi\|$ for all x in $[a, b]$. If x is in $[a, b] - \bigcup_{i=1}^{n_1} I_i$, then

$$\begin{aligned} |G(x) - \phi(x)| &\leq |G(x) - F(x)| + |F(x) - \phi(x)| < \epsilon + |F(x) - \phi(x)| \\ &\leq \epsilon_2 + |F(x) - \phi(x)| \\ &\leq \epsilon_2 + \sup\left\{|F(x) - \phi(x)| : x \text{ in } [a, b] - \bigcup_{i=1}^{n_1} I_i\right\} = \|F - \phi\|. \end{aligned}$$

If x is in I_i for some i $(1 \leq i \leq n_1)$, by definition of G and ϵ_1, respectively, we have

$$-\epsilon < \sigma(-1)^i(G(x) - F(x)) < 0$$

and

$$-\|F - \phi\| + \epsilon_1 \leq \sigma(-1)^i(F(x) - \phi(x)) \leq \|F - \phi\|.$$

By adding these inequalities, we obtain

$$-\|F - \phi\| + (\epsilon_1 - \epsilon) < \sigma(-1)^i(G(x) - \phi(x)) < \|F - \phi\| \quad \text{for all } x \text{ in } I_i,$$

thus $|G(x) - \phi(x)| < \|F - \phi\|$ for x in $\bigcup_{i=1}^{n_1} I_i$. Thus the proof of the lemma is complete.

Remark 2.8. The proof of this lemma does not require the fact that the δ used in Definition 2.3 be allowed to assume the value one. Now, by using the fact that δ may assume the value one, one can prove the following lemma.

LEMMA 2.9. *Let F in $\mathscr{F}$ have a degree $(3, n_2)$ with respect to $\mathscr{F}$ and let ϕ belong to $C[a, b]$. If $F - \phi$ alternates once but does not alternate twice, then there exist G in $\mathscr{F}$ such that $\|G - \phi\| < \|F - \phi\|$.*

COROLLARY 2.10. *Let F in $\mathscr{F}$ have a degree (n_1, n_2) with respect to $\mathscr{F}$ $(n_1 \geq 2)$ and let ϕ belong to $C[a, b]$. If $F - \phi$ alternates once and does not alternate n_1 times then there exist G in $\mathscr{F}$ such that $\|G - \phi\| < \|F - \phi\|$.*

The proof follows from Lemmas 2.7, 2.9, and Corollary 2.6.

LEMMA 2.11. *Let F in $\mathscr{F}$ have a degree $(3, n_2)$ with respect to $\mathscr{F}$ and let ϕ belong to $C[a, b]$. If $F - \phi$ is not a constant function and $F - \phi$ does not alternate once, then there is a function, G, in $\mathscr{F}$ such that $\|G - \phi\| < \|F - \phi\|$.*

Proof. Let $(a_1, b_1) \subset [a, b]$ such that $|F(x) - \phi(x)| < \|F - \phi\|$ for all x in $[a_1, b_1]$. It is noted that a nonempty (a_1, b_1) exists since $F - \phi$ is not a constant function. Define $I_1 = [a, a_1]$, $I_2 = [a_1 + \frac{1}{3}(b_1 - a_1), b_1 - \frac{1}{3}(b_1 - a_1)]$, $I_3 = [b_1, b]$, let σ be in $\{-1, 1\}$ such that there is a y in $[a, b]$ with $\sigma(-1)(F(y) - \phi(y)) = \|F - \phi\|$ (σ is well defined since $F - \phi$ does not alternate once). Furthermore, define ϵ_1 and ϵ_2 as $\epsilon_1 = \min\{\|F - \phi\| + \sigma(-1)(F(x) - \phi(x)): x \text{ in } I_1 \cup I_3\}$, $\epsilon_2 = \|F - \phi\| - \sup\{|F(x) - \phi(x)|: x \text{ in } (a_1, b_1)\}$. It is noted that ϵ_1 and ϵ_2 are positive. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Since $(3, n_2)$ is a degree of F, there exist G in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on I_i $(i = 1, 2, 3)$.

Now we show that $\|G - \phi\| < \|F - \phi\|$. It suffices to show that $|G(x) - \phi(x)| < \|F - \phi\|$ for all x in $[a, b]$. If x belongs to (a_1, b_1), we have $|G(x) - \phi(x)| \leqslant |G(x) - F(x)| + |F(x) - \phi(x)| < \epsilon + |F(x) - \phi(x)| \leqslant \epsilon_2 + |F(x) - \phi(x)| \leqslant \epsilon_2 + \sup\{|F(x) - \phi(x)|: x \text{ in } (a_1, b_1)\} = \|F - \phi\|$. If x belongs to I_i for $i = 1$ or $i = 3$, we have

$$-\epsilon < \sigma(-1)^i(G(x) - F(x)) = \sigma(-1)(G(x) - F(x)) < 0$$

and

$$-\|F - \phi\| + \epsilon_1 \leqslant \sigma(-1)(F(x) - \phi(x)) \leqslant \|F - \phi\|.$$

Thus adding the above two lines gives

$$-\|F - \phi\| + (\epsilon_1 - \epsilon) < \sigma(-1)(G(x) - \phi(x)) < \|F - \phi\|$$

or $|G(x) - \phi(x)| < \|F - \phi\|$ for all x in $I_1 \cup I_3$. Since $|G(x) - \phi(x)| < \|F - \phi\|$ for all x in $[a, b]$, the proof of the lemma is complete.

LEMMA 2.12. *Let F in $\mathscr{F}$ have a degree (n_1, n_2) with respect to $\mathscr{F}$ ($n_1 = 1$ or $n_1 = 2$) and let ϕ belong to $C[a, b]$. If F and ϕ are not identical on $[a, b]$ and if $F - \phi$ does not alternate once then there exist G in $\mathscr{F}$ such that $\|G - \phi\| < \|F - \phi\|$.*

Proof. Let $\epsilon = \min\{\|F - \phi\| + \sigma(-1)(F(x) - \phi(x)): x \text{ in } [a, b]\}$ where σ belongs to $\{-1, 1\}$ such that for some y in $[a, b]$, $\sigma(-1)(F(y) - \phi(y)) = \|F - \phi\|$. Since $(1, n_2)$ is a degree of F (use Lemma 2.5 if $n_1 = 2$), there exist G in $\mathscr{F}$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)(F(x) - G(x)) > 0$ on $[a, b]$. It is easily shown that G is the desired function.

We now give a necessary condition for F with a degree (n_1, n_2) with respect to $\mathscr{F}$ to be a best approximation from $\mathscr{F}$ to a continuous function on $[a, b]$.

THEOREM 2.13. *Let F in $\mathscr{F}$ have a degree (n_1, n_2) with respect to $\mathscr{F}$ and let ϕ belong to $C[a, b]$. If $\|F - \phi\| \leqslant \|G - \phi\|$ for all G in $\mathscr{F}$, then $F - \phi$ alternates at least n_1 times or $F - \phi$ is a constant function.*

Proof. If $n_1 = 0$, then by continuity of $F - \phi$ on $[a, b]$, there is an x in $[a, b]$ such that $|F(x) - \phi(x)| = \|F - \phi\|$.

If $n_1 \geq 1$, assume $F - \phi$ is not a constant function, and $F - \phi$ does not alternate n_1 times; then the previous lemmas insure the existence of a G in $\mathscr{F}$ such that $\|G - \phi\| < \|F - \phi\|$, which is a contradiction.

Remark 2.14. The conclusion in Theorem 2.13 can be made stronger by using Lemma 2.12. That is, if F is in $\mathscr{F}$ and F has $(1, n_2)$ or $(2, n_2)$ as a degree with respect to $\mathscr{F}$, then $\|F - \phi\| \leq \|G - \phi\|$ for all G in $\mathscr{F}$ implies that the error function $F - \phi$ cannot be a nonzero constant on $[a, b]$.

If Definition 2.3 were weakened by requiring δ to be zero, then an alternation theorem weaker than Theorem 2.13 would follow, e.g.,

EXAMPLE 2.15. Let $\mathscr{F}$ denote the set of all functions in $C[-1, 1]$ of the form $c(x - x_1)(x - x_2)(x - x_3)$ where c is an arbitrary real constant and the x_i's are arbitrarily chosen such that $-1 < x_1 < x_2 < x_3 < 1$. The zero function has $(0, 4)$ as a degree. The only property that the zero function is lacking that prevents it from having both $(2, 4)$ and $(4, 4)$ as degrees is that the δ in Definition 2.3 may not assume the value one. As Remark 2.8 indicates if ϕ alternates three times but does not alternate four times, then there is a G in $\mathscr{F}$ such that $\|G - \phi\| < \|\phi\|$. By a similar argument, if ϕ alternates once but not twice, there is a G in $\mathscr{F}$ such that $\|G - \phi\| < \|\phi\|$. If, from this family, zero is the best approximate to a function ϕ, then the maximum number of alternations of ϕ must be either 0, 2, 4, or more. This is illustrated by the functions 1, $2x - 1$, and $8x^4 - 8x^2 - 1$, each of which have the zero function as a best approximation from $\mathscr{F}$, and they alternate a maximum of zero, two, and four times, respectively, with respect to the zero function.

The following theorem gives a sufficient condition for the function F with a degree (n_1, n_2) with respect to $\mathscr{F}$ to be a best approximation to ϕ in $C[a, b]$.

THEOREM 2.16. *Let F belong to $\mathscr{F}$ and let ϕ belong to $C[a, b]$. If $F - \phi$ alternates n_2 times in $[a, b]$ and if F has a degree (n_1, n_2) with respect to $\mathscr{F}$, then $\|F - \phi\| \leq \|G - \phi\|$ for all G in $\mathscr{F}$.*

Proof. Suppose G is in $C[a, b]$ such that $\|F - \phi\| > \|G - \phi\|$. Let $\{x_i\}_{i=1}^{n_2+1}$ be a set of alternation points for the function $F - \phi$. Let σ be in $\{-1, 1\}$ such that $\sigma(-1)(F(x_1) - \phi(x_1)) = \|F - \phi\|$, then $\sigma(-1)^i \times (F(x_i) - \phi(x_i)) = \|F - \phi\|$ $(i = 1,..., n_2 + 1)$. Since $\sigma(-1)^i(F(x_i) - G(x_i)) = \sigma(-1)^i(F(x_i) - \phi(x_i)) + \sigma(-1)^i(\phi(x_i) - G(x_i)) = \|F - \phi\| + \sigma(-1)^i(\phi(x_i) - G(x_i))$ is positive $(i = 1,..., n_2 + 1)$, we have $\sigma^2(-1)^{2i+1} \times (F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) > 0$ $(i = 1,..., n_2)$ or $(F(x_i) - G(x_i)) \times (F(x_{i+1}) - G(x_{i+1})) < 0$ $(i = 1,..., n_2)$. Since (n_1, n_2) is a degree of F, the last inequality implies that G is not in $\mathscr{F}$. Therefore, there is no function, G, in $\mathscr{F}$ where $\|F - \phi\| > \|G - \phi\|$.

Rice, in his thesis, used the term varisolvent family to describe his family of approximating functions. Since our definition extends the ideas of Rice, at the risk of confusion, we also call our approximating families varisolvent.

DEFINITION 2.17. Let $\mathscr{F}$ be a nonempty family of functions in $C[a, b]$. $\mathscr{F}$ will be called a *varisolvent family of functions* if every function, F, in $\mathscr{F}$ has a degree with respect to $\mathscr{F}$. (We show later that a family that is varisolvent in the sense of Rice is also a varisolvent family in the above sense.)

From the above discussion we have the following alternation theorem for varisolvent families.

THEOREM 2.18. *Let $\mathscr{F}$ be a varisolvent family of functions on the interval $[a, b]$. Let F in $\mathscr{F}$ have a degree (n_1, n_2) with respect to $\mathscr{F}$, and let ϕ belong to $C[a, b]$.*

(1) *If $\| F - \phi \| \leqslant \| G - \phi \|$ for all G in $\mathscr{F}$, then either $F - \phi$ is a constant or $F - \phi$ alternates n_1 times on $[a, b]$.*

(2) *If $F - \phi$ alternates n_2 times on $[a, b]$ then $\| F - \phi \| \leqslant \| G - \phi \|$ for all G in $\mathscr{F}$.*

3. EXAMPLES

A. *Haar System*

DEFINITION 3.1. Let $\mathscr{F}$ be an n-dmiensional subspace of $C[a, b]$ $(n \geqslant 1)$. The set $\mathscr{F}$ is an *n-dimensional Haar system* if for every set of n distinct points $\{x_i\}_{i=1}^{n}$ in $[a, b]$ and for any set of n real numbers $\{y_i\}_{i=1}^{n}$, there is a unique element F in $\mathscr{F}$ such that $F(x_i) = y_i$ $(i = 1,..., n)$.

Let $\mathscr{F}$ be an n-dimensional Haar system and $F \in \mathscr{F}$. It can be verified that F has degree (n, n), i.e., $n_1 = n$ and $n_2 = n$ where n_1 and n_2 are as given in Definition 2.3.

Further, it has been shown [1] that every Haar System, $\mathscr{F}$, of dimension n $(n \geqslant 1)$ on $[a, b]$ has a function which is positive on the whole interval. Therefore, every function in $\mathscr{F}$ has $(1, n)$ and (n, n) as degrees. The classical alternation theorem will follow from Theorem 2.18 and Remark 2.14, that is, if F belongs to $\mathscr{F}$ and ϕ belongs to $C[a, b]$ where ϕ is not identical to F, then F is a best approximation to ϕ from $\mathscr{F}$ if and only if $F - \phi$ alternates n times.

B. *Weak Chebyshev System*

DEFINITION 3.2. Let $\mathscr{F}$ be an n-dimensional subspace of $C[a, b]$. The set $\mathscr{F}$ is an *n-dimensional weak Chebyshev system* if every function F in $\mathscr{F}$ has at most $n - 1$ zero crossings (that is, if G is in $C[a, b]$ and if $\{x_i\}_{i=1}^{n}$ is an increasing set of points in $[a, b]$ such that $G(x_i)\, G(x_{i+1}) < 0$ ($i = 1,\dots, n-1$) then G is not in $\mathscr{F}$).

Remark 3.3. If $\mathscr{F}$ is an n-dimensional weak Chebyshev system of $C[a, b]$, then every F in $\mathscr{F}$ has a degree of $(0, n)$.

As a special case of a weak Chebyshev system, we have the polynomial spline functions.

Remark 3.4. If $\mathscr{F} = S_{n,k}(x_1, \dots, x_k)$ for $n \geq 1$, the polynomial spline functions with knots at $\{x_i\}_{i=1}^{k}$ with $a < x_i < x_{i+1} < b$ $(1 \leq i < k-1)$ (that is, $\mathscr{F}$ is the linear span of $\{1, \dots, x^n, (x - x_1)_+^n, \dots, (x - x_k)_+^n\}$ where $(t)_+^n = t^n$ for $t \geq 0$ and $(t)_+^n = 0$ for $t < 0$), then F in $\mathscr{F}$ has a degree of $(n + 1, n + 1 + k)$.

The proof of this requires the following observation.

Remark 3.5. Let $\mathscr{F}$ be a subset of $C[a, b]$ and let F in $\mathscr{F}$ have (n_1, n_2) as a degree with respect to $\mathscr{F}$. Let $\mathscr{F}_1$ be a subset of $\mathscr{F}$ with F belonging to $\mathscr{F}_1$. If F has (m_1, m_2) as a degree with respect to $\mathscr{F}_1$, then (m_1, n_2) is also a degree of F with respect to $\mathscr{F}$.

C. *Varisolvent Family in the Sense of Rice*

We will now show that a varisolvent family in the sense of Rice is in fact a varisolvent family as defined by Definition 2.17.

DEFINITION 3.6. Let $\mathscr{F}$ be a subset of $C[a, b]$. The set $\mathscr{F}$ is a *varisolvent family in the sense of Rice* on $[a, b]$ if for every F in $\mathscr{F}$, there is an integer $n(F) \geq 1$ such that the following two conditions are satisfied:

(1) Let $\{x_i\}_{i=1}^{n(F)}$ be an arbitrary set of $n(F)$ distinct points in $[a, b]$ and let ϵ be an arbitrary positive number. Then there is a $\gamma(F, \epsilon; \{x_i\}_{i=1}^{n(F)}) > 0$ where if $\{y_i\}_{i=1}^{n(F)}$ is a set of real numbers such that $|y_i - F(x_i)| < \gamma$ $(i = 1,\dots, n(F))$, then there is a G in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $G(x_i) = y_i$ $(i = 1,\dots, n(F))$.

(2) If F_1 is in $\mathscr{F}$ and $F(x_i) = F_1(x_i)$ $(i = 1,\dots, n(F))$ where $\{x_i\}_{i=1}^{n(F)} \subset [a, b]$ and the x_i's are distinct, then F and F_1 are identical on $[a, b]$.

The number $n(F)$ will be referred to as the *varisolvency degree of* F.

If $\mathscr{F}$ is a varisolvent family in the sense of Rice on $[a, b]$ and if F in $\mathscr{F}$ has varisolvency degree m, then F has a degree (m, m) with respect to $\mathscr{F}$. The proof of this remark follows from the following remark and the second part of the definition of a varisolvent family in the sense of Rice.

Remark 3.7. By using a zero counting argument (allowing for multiple zeros) the following can be shown. Let $\mathscr{F}$ be a varisolvent family in the sense of Rice on $[a, b]$. Let F in $\mathscr{F}$ have $n(F)$ as the degree of varisolevncy $(n(F) \geq 1)$. Let ϵ be a positive number and let $\sigma \in \{-1, 1\}$ be arbitrarliy chosen. Let δ

be an arbitrary element of $\{0, 1\}$ where $\delta < n(F)$. Let $\{x_i\}_{i=1}^{n(F)+1-\delta}$ be an arbitrary increasing set of points such that $x_1 = a$ and $x_{n(F)+1-\delta} = b$. Then there is a function G in $\mathscr{F}$ with $(F(a) - G(a))(F(b) - G(b)) \neq 0$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on the open interval (x_i, x_{i+1}) $(i = 1,..., n(F) - \delta)$.

D. *Varisolvent Family with Constant Error Phenomenon*

If F belongs to $\mathscr{F}$, $\mathscr{F}$ a varisolvent family in the sense of Rice on $[a, b]$ with m the varisolvency degree of F $(m > 3)$, then, in general, it is not known whether or not there is a G in $\mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(F(x) - G(x)) > 0$ for all x in $[a, b]$ for arbitrary positive ϵ and for arbitrary σ in $\{-1, 1\}$, i.e., whether or not $(1, m)$ is a degree of F. Additional hypotheses have been shown to be sufficient to eliminate the possibility of a nonzero constant error [6]. The following is an example of a family of functions, $\mathscr{F}$, which is varisolvent on $[a, b]$, such that every function, F, in $\mathscr{F}$ has a degree (m, m) with respect to $\mathscr{F}$ for some $m \geqslant 1$, where m depends on F. Further, there is a function F in $\mathscr{F}$ such that $(1, m)$ is not a degree. It will be seen that F is a best approximation to the continuous function $F(x) - 1$ $(a \leqslant x \leqslant b)$ from $\mathscr{F}$, and hence the error function $F(x) - (F(x) - 1)$ is constant.

EXAMPLE 3.8. Let $\mathscr{F}_1$ be a varisolvent family in the sense of Rice on $[a, b]$ which possesses a function, call it F, with degree of varisolvency m, $m \geqslant 3$. Construct the family $\mathscr{F}$ as follows. Let

$$\mathscr{G} = \{G \text{ in } \mathscr{F}_1\text{: for some } a \leqslant z_1 < z_2 \leqslant b, (F(z_1) - G(z_1)(F(z_2) - G(z_2)) < 0\}$$

and

$$\mathscr{F} = \{F\} \cup \mathscr{G}.$$

Claim. The family $\mathscr{G}$ is a varisolvent family in the snese of Rice. Further, if G belongs to $\mathscr{G}$ and if n is its varisolvency degree in $\mathscr{F}_1$, then n is its varisolvency degree in $\mathscr{G}$.

Proof. Let G be an arbitrary function in $\mathscr{G}$. Denote its varisolvency degree in $\mathscr{F}_1$ by n. Let ϵ_1 be a positive number and $\{x_i\}_{i=1}^{n}$ a set of n distinct points in $[a, b]$ be chosen arbitrarily. Since G belongs to $\mathscr{G}$ we have $a \leqslant z_1 < z_2 \leqslant b$ such that $(F(z_1) - G(z_1))(F(z_2) - G(z_2)) < 0$. Let $\epsilon_2 = \min\{| F(z_i) - G(z_i)| := 1, 2\}$ and $\epsilon = \min(\epsilon_1, \epsilon_2)$. Since G belongs to $\mathscr{F}_1$, we have the existence of γ $(G, \epsilon; x_1 ,..., x_n) > 0$ such that if $\{ y_i\}_{i=1}^{n}$ is a set of real numbers such that $| y_i - G(x_i)| < \gamma$ $(i = 1,..., n)$, then there is a function H in $\mathscr{F}_1$ where $\| G - H \| < \epsilon$ and $H(x_i) = y_i$ $(i = 1,..., n)$. We will show that H is also in $\mathscr{G}$. Since $| F(z_i) - G(z_i)| \geqslant \epsilon_2 \geqslant \epsilon > | G(z_i) - H(z_i)|$ $(i = 1, 2)$ we have $(F(z_i) - G(z_i))^2 = | F(z_i) - G(z_i)|^2 = |(G(z_i) - H(z_i)) \times (F(z_i) - G(z_i))|$ $(i = 1, 2)$ or $(F(z_i) - G(z_i))^2 - (H(z_i) - G(z_i))(F(z_i) -$

$G(z_i)) = (F(z_i) - H(z_i))(F(z_i) - G(z_i)) > 0$ ($i = 1, 2$). Now $(F(z_1) - H(z_1)) \times (F(z_2) - H(z_2))(F(z_1) - G(z_1))(F(z_2) - G(z_2)) > 0$ and $(F(z_1) - G(z_1)) \times (F(z_2) - G(z_2)) < 0$ imply that $(F(z_1) - H(z_1))(F(z_2) - H(z_2)) < 0$. Therefore H is in $\mathscr{G}$ and thus G has varisolvency degree n in $\mathscr{G}$ completing the proof.

From the discussion in Section C above, it follows that if n is the degree of varisolvency of G in $\mathscr{G}$, then a degree of G in $\mathscr{G}$ with respect to $\mathscr{G}$ is (n, n). It also follows from Remark 3.5 that (n, n) is a degree of G with respect to $\mathscr{F}$.

Claim. F has degree (m, m) with respect to $\mathscr{F}$.

Proof. (1) Let $\epsilon > 0$, σ in $\{-1, 1\}$, δ in $\{0, 1\}$ be arbitrary, and let $\{[c_i, d_i]\}_{i=1}^{m-\delta}$ be an arbitrary increasing sequence of intervals with $c_1 = a$ and $d_{m-\delta} = b$. Since $\mathscr{F}_1$ is a varisolvent family as shown above, there is a function G in $\mathscr{F}_1$ such that $\|F - G\| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ ($i = 1,\dots, m - \delta$). Since $m - \delta \geq 2$ we have G belongs to $\mathscr{G}$.

(2) Let G belong to $C[a, b]$ and let $\{x_i\}_{i=1}^{m+1}$ be a subset of $[a, b]$ with $x_i < x_{i+1}$ ($i = 1,\dots, m$) such that $(F(x_i) - G(x_i))(F(x_{i+1}) - G(x_{i+1})) < 0$ ($i = 1,\dots, m$). Then, since F has a degree (m, m) with respect to $\mathscr{F}_1$, G does not belong to $\mathscr{F}_1$, and hence G does not belong to $\mathscr{F}$. This completes the proof of the claim.

From the construction of $\mathscr{F}$, it is further noted that F does not have $(1, m)$ as a degree with respect to $\mathscr{F}$. It is clear from the construction of $\mathscr{F}$ that the function F in $\mathscr{F}$ is a best approximation to the continuous function $F(x) - 1$ ($a \leq x \leq b$) from $\mathscr{F}$, and hence the error function $F(x) - (F(x) - 1)$ is constant. We reemphasize the fact that it is not known whether a varisolvent family in the sense of Rice permits a constant error function.

E. *Another Example*

A partly alternating family, $\mathscr{F}$, as defined by Dunham [5], is a special case of a varisolvent family. If F belongs to $\mathscr{F}$, then there exists an $m \geq 1$ such that (m, m) and $(1, m)$ are degrees of F with respect to $\mathscr{F}$.

F. *Generalized Exponential Sums*

Let R^n denote the n-dimensional linear space of real numbers, and let $\|\cdot\|$ denote some norm defined on R^n. (By the context, there is no confusion between $\|\cdot\|$ defined on R^n and $\|\cdot\|$ defined on $C[a, b]$.) The symbol $\mathbf{a}$ denotes an element of R^n, where $\mathbf{a} = (a_1, \dots, a_n)$. If $\mathbf{a}, \mathbf{b}$ are in R^n then $\mathbf{a} \cdot \mathbf{b}$ denotes the usual dot product, that is, $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$. If $F(\mathbf{a}; x)$ is a function of x ($a \leq x \leq b$) which depends on the parameter $\mathbf{a}$, then with sufficient smoothness assumptions, the gradient of F at $\mathbf{b}$ is defined by $(\text{grad } F)(\mathbf{b}: x) = ((\partial F/\partial a_1)(\mathbf{b}: x), \dots, (\partial F/\partial a_n)(\mathbf{b}: x))$.

The following lemma relates a local Haar type property to a degree of a function with respect to a family of continuous real-valued functions. Its proof is omitted.

LEMMA 3.9. *Let A be an open subset of R^n. Let F be a mapping from A into $C[a, b]$ (F: $\mathbf{a} \to F(\mathbf{a}; x)$ $(a \leqslant x \leqslant b)$). Let $\mathscr{F} = \{F(\mathbf{a})$: $\mathbf{a}$ in $A\}$. Suppose for a particular $\mathbf{a}$ in A, the function of x $F(\mathbf{a})$ has (n_1, n_2) as a degree with respect to $\mathscr{F}$. Further, assume* (grad F)($\mathbf{a}$; x) *exists and is continuous as a function of x. Let $r(\mathbf{a}, \mathbf{b} - \mathbf{a}; x) = F(\mathbf{b}; x) - F(\mathbf{a}; x) - (\mathbf{b} - \mathbf{a}) \cdot (\operatorname{grad} F)(\mathbf{a}; x)$ ($\mathbf{b}$ in A). If the zero function (as a function of x) has a degree (m_1, m_2) with respect to the family of continuous functions of x in $\{\mathbf{b} \cdot (\operatorname{grad} F)(\mathbf{a}; x)$: $a \leqslant x \leqslant b$, $\mathbf{b}$ in $R^n\}$, and if $\| r(\mathbf{a}, \mathbf{b} - \mathbf{a})\| = o(\| \mathbf{b} - \mathbf{a} \|)$, then (m_1, n_2) is a degree of $F(a)$ with respect to $\mathscr{F}$.*

DEFINITION 3.10. Let $n \geqslant 1$. Let the set of *generalized exponential sums*, E_n, *of degree n* be defined as functions of x on the interval $[a, b]$ as follows.

$$E_n = \Big\{ \sum_{i=1}^{l} \sum_{j=0}^{m_i} a_{ij} x^j \exp(\alpha_i x) \colon a_{ij}\text{'s and } \alpha_i\text{'s are real numbers;}$$

$$a_{ij} = 0 \; (j = 0, \ldots, m_i \,;\, i = 1, \ldots, l) \text{ or } \alpha_i < \alpha_{i+1} \; (1 \leqslant i < l),$$

$$a_{im_i} \neq 0 \; (i = 1, \ldots, l) \text{ and } \sum_{i=1}^{l} (m_i + 1) \leqslant n. \Big\}$$

The following theorem is due to Polya–Szego [7] (e.g., see [9]).

THEOREM 3.11. *Every F in E_n has at most $n - 1$ zeros or vanishes identically.*

THEOREM 3.12. *For every F in E_n, F has $(1, 2n)$ as a degree with respect to E_n.*

Proof. Let F belong to E_n.

(1) Let positive ϵ and σ in $\{-1, 1\}$ be arbitrarily chosen. If $F(x) = 0$ for all x in $[a, b]$, then let $G(x) = \frac{1}{2}\sigma\epsilon$ $(a \leqslant x \leqslant b)$. Then G belongs to E_n, $\sigma(-1)(F(x) - G(x)) = \epsilon/2 > 0$ and $\| F - G \| = \epsilon/2 < \epsilon$. If $F(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i} a_{ij} x^j \exp(\alpha_i x)$, define G in E_n such that $G(x) = F(x) + (\sigma\epsilon/2)(\exp(\alpha_1 x)/\| \exp(\alpha_1 x)\|)$. Then $\| F - G \| = \epsilon/2$ and $\sigma(-1)(F(x) - G(x)) = (\epsilon/2)(\exp(\alpha_1 x)/\| \exp(\alpha_1 x)\|) > 0$.

(2) Let G belong to E_n, then $F - G$ belongs to E_{2n}. According to the theorem above, $F - G$ has at most $2n - 1$ zeros, or $F - G = 0$ for all x in $[a, b]$. This completes the proof of the lemma.

It is noted that since every element of E_n has a degree with respect to E_n, we have that E_n is a varisolvent family on $[a, b]$.

THEOREM 3.13. *The zero function has* (n, n) *as a degree with respect to* E_n.

Proof. Let $\mathscr{F}_1 = \{\sum_{i=0}^{n-1} a_i x^i : a_i$ a real numbers, $i = 0, \ldots, n-1\} \subset E_n$. The zero function is in $\mathscr{F}_1$ so it has (n, n) for a degree with respect to $\mathscr{F}_1 \subset E_n$. Therefore, according to Remark 3.5, $(n, 2n)$ is a degree of the zero function with respect to E_n. It then follows from Theorem 3.16 that (n, n) is a degree of the zero function with respect to E_n.

The proof of the following theorem is lengthy and is omitted.

THEOREM 3.14. *Let* F *belong to* E_n *where* $F(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i} a_{ij} x^j \exp(\alpha_i x)$ *with* $\alpha_i < \alpha_{i+1}$ $(1 \leq i < l)$, $a_{im_i} \neq 0$ $(i = 1, \ldots, l)$ $k = \sum_{i=1}^{l} (m_i + 1) \leq n$. *Then* F *has* $(n + l, n + k)$ *as a degree with respect to* E_n.

The following corollary to Theorem 3.14 appeared as Theorem 4.2 in Werner [9].

COROLLARY 3.15. *Let* F *be a function in* E_n $(n \geq 1)$ *on* $[a, b]$. *If* $F \equiv 0$ *on* $[a, b]$, *let* $l = k = 0$. *If* F *is not the zero function let* $F(x) = \sum_{i=1}^{l} \sum_{j=0}^{m_i} a_{ij} x^j \exp(\alpha_i x)$ *with* $l \geq 1$, $\alpha_i < \alpha_{i+1}$ $(1 \leq i < l)$, $m_i \geq 0$ $(i = 1, \ldots, l)$, $a_{im_i} \neq 0$ $(i = 1, \ldots, l)$, *and* $k = \sum_{i=1}^{l} (m_i + 1) \leq n$. *Let* ϕ *belong to* $C[a, b] - E_n$.

(1) *If* $\|F - \phi\| \leq \|G - \phi\|$ *for all* G *in* E_n, *then* $F - \phi$ *alternates at least* $n + l$ *times on* $[a, b]$.

(2) *If* $F - \phi$ *alternates* $n + k$ *times on* $[a, b]$, *then* $\|F - \phi\| \leq \|G - \phi\|$ *for all* G *in* E_n.

Proof. By Theorem 3.14 (Theorem 3.13 if F is identically zero) and Lemma 3.12, respectively, it follows that $(n + l, n + k)$ and $(1, 2n)$ are degrees of F with respect to E_n. An application of the alternation theorem, Theorem 2.18, and Remark 2.14 completes the proof.

G. *Miscellaneous Examples*

Remark 3.16. Let $\mathscr{F}$ be an n-dimensional Haar system on $[a, b]$, let $\{F_i\}_{i=1}^{n}$ be a basis for $\mathscr{F}$, and let A be a subset of R^n. The family, $\mathscr{F}_1$, defined such that $\mathscr{F}_1 = \{\sum_{i=1}^{n} a_i F_i : (a_1, \ldots, a_n)$ in $A\}$ is a varisolvent family. If F belongs to $\mathscr{F}_1$, then $(0, n)$ is a degree of F with respect to $\mathscr{F}_1$. If F belongs to $\mathscr{F}_1$ such that $F = \sum_{i=1}^{n} a_i F_i$ and $(a_1, \ldots, a_n)$ is in the interior of A, then $(1, n)$ and (n, n) are degrees of F with respect to $\mathscr{F}_1$.

Remark 3.17. Let γ be a mapping from $A \times [a, b] \subset R^2$ into R $(\gamma : (\alpha, x) \to \gamma(\alpha, x))$ such that $(\partial/\partial\alpha)^j \gamma(\beta, \cdot)$ exists and belongs to $C[a, b]$ for all β in A $(j = 0, 1, \ldots)$. Furthermore, suppose for all $l \geq 1$, $m_i \geq 0$

$(i = 1,..., l)$ and α_i in A $(i = 1,..., l)$, the subspace of $C[a, b]$, $\langle\{\bigcup_{i=1}^{l} \bigcup_{j=0}^{m_i} (\partial/\partial\alpha)^j \gamma(\alpha_i, x)\}\rangle$ is a $\sum_{i=1}^{l} (m_i + 1)$-dimensional Haar system. Then, for each $n \geqslant 1$, the family of functions, V_n, of x where $V_n = \{F$ in $C[a, b]$: $F(x) = 0$ $(a \leqslant x \leqslant b)$ or $F(x) = \sum_{i=1}^{l(F)} \sum_{j=0}^{m_i} a_{ij}(\partial/\partial\alpha)^j(\alpha_i, x)$ where $(F) \geqslant 1$, $m_i \geqslant 0$ $(i = 1,..., l(F))$, a_{ij}'s are real, α_i's belong to A and are distinct, $a_{im_i} \neq 0$ $(i = 1,..., l(F))$ and $k(F) = \sum_{i=1}^{l(F)} (m_i + 1) \leqslant n\}$ is a varisolvent family on $[a, b]$ in the sense of Definition 2.17. Further, if F belongs to V_n, then $(n + l(F), n + k(F))$ and $(1, n + k(F))$ are degrees of F with respect to V_n, where $l(F) = k(F) = 0$ if $F(x) = 0$ $(a \leqslant x \leqslant b)$. In particular, the above is applicable if $A = \{\alpha$ in R: $0 < |\alpha| < 1\}$, $[a, b] = [-1, 1]$ and $(\alpha, x) = (1 + \alpha x)^{-1}$. The class of families, V_n, described above is a special of γ-polynomials (e.g., see [3]).

Remark 3.18. It follows from the previous remark that if $A = \{\alpha$ in R: $0 < |\alpha| < 1\}$ and $[a, b] = [-1, 1]$, then $L_n = \{F$ in $C[a, b]$: $F(x) = c$, where c is a constant, $(-1 \leqslant x \leqslant 1)$ or $F(x) = a_{00} + \sum_{i=1}^{l(F)} \sum_{j=0}^{m_i} a_{ij}(\partial/\partial\alpha)^j \log(1 + \alpha_i x)$ where $l(F) \geqslant 1$, $m_i \geqslant 0$ $(i = 1,..., l(F))$, a_{ij}'s are real, α_i's belong to A and are distinct, $a_{im_i} \neq 0$ $(i = 1,..., l(F))$ and $k(F) = \sum_{i=1}^{l(F)} (m_1 + 1) \leqslant n\}$ is a varisolvent family in the sense of Definition 2.17. Further, if F belongs to L_n, then $(n + l(F) + 1, n + k(F) + 1)$ and $(1, n + k(F) + 1)$ are degrees of F with respect to L_n where $l(F) = k(F) = 0$ whenever F is identically equal to a constant. (See [4] for a discussion when $l(F) = 1$ and $m_1 = 0$.)

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